The radius of starlikeness for some analytic functions

By

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Abstract

Let $F(z)$ be reqular in $|z|<1$ and $F(0)=0$, $F'(0)=1$. Let $f(z)=1/(c+1)z^{1-c}\{z^cF(z)\}'$ and c be a real and non-negative constant. we have determined the radius of starlikeness of order β for $f(z)$ when $F(z)$ is starlike of order $\alpha, 0\leqq\beta, \alpha<1$.

1, Introduction

In this paper we shall treat a generalization of the recent results of H. S. Al-Amri [1]. The method used in the proof of the theorem is that of V.A. Zmorovič [5]. The class $S^*(\alpha)$ is called the starlike functions of order α. Analytically, $F(z)\in S^*(\alpha)$ if and only if $\mathrm{Re}\{zF'(z)/F(z)\}>\alpha$ in $|z|<1$.
We shall denote by P the class of regular functions $p(z)$ in $|z|<1, p(0)=1$ such that $\mathrm{Re}\{p(z)\}>0$. Let

(1) $$f(z)=\frac{1}{c+1}z^{1-c}[z^cF(z)]',$$ and c be a real and non-negative constant.

Let $F(z)=(c+1)z^{-c}\int_0^z t^{c-1}f(t)dt$ be regular in $|z|<1$ and $F(0)=0$, $F'(0)=1$.
If $F(z)\in S^*(\alpha)$, then there exists $p(z)\in P$ such that

(2) $$zF'(z)/F(z)=\alpha+(1-\alpha)p(z).$$

From (1), (2) we get
$$z^cf(z)/\int_0^z t^{c-1}f(t)dt=c+\alpha+(1-\alpha)p(z).$$
Thus
$$zf'(z)/f(z)-\beta$$
(3) $$=-(c+\beta)+(1-\alpha)(h+p(z))+zp'(z)/(h+p(z)),$$
Where $h=(c+\alpha)/(1-\alpha), 0\leqq\alpha<1$.

We shall determine $r_{\alpha,\beta,c}$ the radius of starlikeness of order β for $f(z)$. Clearly, $r_{\alpha,\beta,c}$ is the smallest positive root of $Q_{\alpha,\beta,c}=0$, where

(4) $$Q_{\alpha,\beta,c}(r)=\min_{p\in P}\min_{|z|=r<1}\mathrm{Re}\{-(c+\beta)+(1-\alpha)(h+p(z))$$

$+ zp(z)/(h+p(z))\}.$

Thus this problem are reduced to seeking the guantity

(5) $Q(r) = \min_{p\in P} \min_{|z|=r<1} \mathrm{Re}\{\Psi(p(z),\ zp'(z))\},$

where $\Psi(w\,;W)$ is an analytic function of the variables w and W in the W-plane and in the half plane $\mathrm{Re}\{w\}>0$.

By M. S. Robertson's variational method [4], the minimum in (5) is realized for functions of the form

(6) $$p(z)=\lambda_1\frac{1+ze^{-i\theta_1}}{1-ze^{-i\theta_1}}+\lambda_2\frac{1+ze^{-i\theta_2}}{1-ze^{-i\theta_2}}$$

where $\theta_1, \theta_2 \in [0, 2\pi]$, $\lambda_1, \lambda_2 \geqq 0$ and $\lambda_1+\lambda_2 \geqq 0$ and $\lambda_1+\lambda_2=1$.

We obtain $r_{\alpha,\beta,c}$ by an application of a theorem due to V. A. Zmorovič.

Theorem A (V. A. Zmorovič). Let $\Psi(w\,;W) = M(w)+N(w)W$, where $M(w)$ and $N(w)$ are defined and finite in the half plane $\mathrm{Re}\{w\}>0$. Put

$$w=\lambda_1\frac{1+z_1^m}{1-z_1^m}+\lambda_2\frac{1+z_2^m}{1-z_2^m}$$

$$W=\lambda_1\frac{2mz_1^m}{(1-z_1^m)^2}+\lambda_2\frac{2mz_2^m}{(1-z_2^m)^2},$$

where z_1 and z_2 are arbitrary points on $|z|=r<1$, m is a positive integer, $\lambda_1 \geqq 0$, $\lambda_2 \geqq 0$, $\lambda_1+\lambda_2=1$.

Then the function $\Psi(w, W)$ has the form

$$\Psi(w\,;W)=M(w)+\frac{1}{2}m(w^2-1)N(w)+\frac{1}{2}m(\rho^2-\rho_0^2)N(w)e^{2i\Psi},$$

where

$$(1+z_k^m)/(1-z_k^m) = a+\rho\exp(i\psi_k) \qquad (k=1,2),$$

$$w=a+\rho_0\exp(i\psi_0) \qquad (0\leqq\rho_0\leqq\rho),$$

$|z_1| = |z_2| = r$, $a=(1+r^{2m})/(1-r^{2m})$,

$\rho=2r^m/(1-r^{2m})$, $e^{i\psi}=i\exp[i(\psi_1+\psi_2)/2]$.

Also

(7) $$\min \mathrm{Re}\{\Psi(w\,;W)\} \equiv \Psi_\rho(w)$$

$$= \mathrm{Re}\{M(w)+\frac{1}{2}m(w^2-1)N(w)\}-\frac{1}{2}m|N(w)|(\rho^2-\rho_0^2).$$

The minimun in (7) is reached when

(8) $$\exp[i(2\psi+\arg N(w))] = -1.$$

Comparing problem (4) with the terms of Theorem A, we have $m=1$, and

$$M(w) = -(c+\beta) + (1-\alpha)(w+h),$$
$$N(w) = 1/(w+h),$$
$$h = (c+\alpha)/(1-\alpha), \qquad 0 \leqq \alpha < 1.$$

From (4), (7) and these relations our problem is reduced to minimizing $\Psi_\rho(w)$, where

(9) $$\Psi_\rho(w) = \mathrm{Re}\left\{-(c+\beta) + (1-\alpha)(w+h) + \frac{1}{2}\frac{w^2-1}{w+h}\right\} - \frac{1}{2}\frac{\rho^2-\rho_0^2}{|w+h|}$$

2, Radius of starlikeness

Theorem. Let $F(z) \in S^*(\alpha)$, $0 \leqq \alpha < 1$,
$f(z) = \frac{1}{c+1} z^{1-c}[z^c F(z)]'$ in $|z| < 1$. Let $r_{\alpha,\beta,c}$ be the radius of the largest disk in which $\mathrm{Re}\{zf'(z)/f(z)\} > \beta$.

Put

(10) $$a^2 + 2(\beta - 2\alpha - c)a + (2h+\beta+c)^2 - 4(2-\alpha)h^2 = 0,$$

(11) $$\{4\alpha^2 - 2(\beta - c + 2)\alpha + (1+\beta)(1-c)\}r^2 + 2\{(3-\beta+c)\alpha - c\beta - 2\}r + (1+c)(1-\beta) = 0,$$

where $a = (1+r^2)/(1-r^2)$, $h = (c+\alpha)/(1-\alpha)$; also

(12) $$B_1\beta^3 + B_2\beta^2 + B_3\beta + B_4 = 0$$

where

$$B_1 = 4(1-\alpha)\{2c + (2-c)\alpha - \alpha^2\},$$
$$B_2 = -4\alpha^4 + 4(c+1)\alpha^3 + (9 - 28c + 8c^2)\alpha^2 + 2(21c - 16c^2)\alpha + 33c^2,$$
$$B_3 = 4(1-c)\alpha^4 + 2(1+12c-c^2)\alpha^3 + (2c^3 + 14c^2 - 23c - 29)\alpha^2 - (6c^3 + 21c^2 + 29c - 14)\alpha + 4c^2 + 14c,$$
$$B_4 = (2\alpha^2 + c^2\alpha + 3c\alpha + 3\alpha - c^2 - 2)^2 + 2(1-\alpha)(2\alpha + c)^2(2\alpha^2 + c^2\alpha + 3c\alpha + 3\alpha - c^2 - 2) + (2\alpha + c)^2(2\alpha - c\alpha + 3c)^2 - 4(2-\alpha)(c+\alpha)^2(2\alpha + c).$$

Then $r_{\alpha,\beta,c}$ is the smallest positive root r in (10) with $0 \leqq \beta \leqq \beta_0(\alpha, c)$ and the smallest positive root r in (11) with $\beta_0(\alpha, c) \leqq \beta < 1$, where $\beta_0(\alpha, c)$ is the smallest positive root of (12). These results are sharp.

proof. Our problem is reduced to minimizing $\psi_\rho(w)$. This minimum is reached when the point w ($|w-a| < \rho$) is fixed, and the chord passing through it and through the points $a + \rho \exp i\psi_k$ ($k = 1,2$) is perpendicular to the vector $\exp i\phi/2$, where $w + h = R \exp i\phi$.

By setting $w = a + \xi + i\eta$, $\rho_0^2 = \xi^2 + \eta^2 \leqq \rho^2$.

Then (9) becomes

$$\psi_\rho(w) \equiv \psi_\rho(\xi;\ \eta)$$

(13) $$= -(c+\beta+h) + \left(\frac{3}{2}-\alpha\right)(a+\xi+h) + \frac{1}{2}(h^2-1)(a+\xi+h)R^{-2} - \frac{1}{2}(\rho^2-\xi^2-\eta^2)R^{-1},$$

where $R^2 = (a+\xi+h)^2+\eta^2$.

We now wish to minimize $\psi_\rho(w)$ as a function of η. A differentiation shows that

(14) $$\partial\psi_\rho/\partial\eta = \frac{1}{2}\eta R^{-4} S(\xi,\ \eta),$$

where

(15) $$S(\xi,\ \eta) = [\xi^2+4(a+h)\xi+\rho^2+\eta^2+2(a+h^2)]R - 2(h^2-1)(a+\xi+h)$$
$$\geqq [\xi^2+4(a+h)\xi+\rho^2+2(a+h)^2-2(h^2-1)]\ (a+\xi+h).$$

But the last expression in (15) is an increasing function of ξ in the interval $[-\rho,\ \rho]$. Hence

$$S(\xi,\ \eta) \geqq S(-\rho,\ \eta) = 2[(a-\rho)^2+2h(a-\rho)+1](a+h-\rho) > 0.$$

Thus we see from (14) that $\Psi_\rho(\xi,\ \eta)$ is minimized on every chord $\xi =$ constant of the circle $\xi^2+\eta^2=\rho_0^2$ at the point $\eta=0$.

Therefore the minimum of $\psi_\rho(\xi;\ \eta)$ in the disk $\xi^2+\eta^2 \leqq \rho^2$ occurs somewhere on the diameter $\eta=0$. Setting $\eta=0$ in (13), we have

(16) $\psi_\rho(\xi;\ 0) \equiv \ell(R) = (2-\alpha)R + (h^2+ah)R^{-1} - (a+2h) - (c+\beta)$.

The absolute minimum of $\ell(R)$ is realized at

(17) $$R_0 = [(h^2+ah)/(2-\alpha)]^{1/2}$$

where $R_0 < a+h+\rho$.

However, if $R_0 \notin [a+h-\rho,\ a+h+\rho]$, then the minimum of $\ell(R)$ is attained at

(18) $$R_1 = a+h-\rho.$$

The radius $r_{\alpha,\beta,c}$ is therefore determined from either

(19) $$\ell(R_0) = 0,$$

where R_0 is given by (17), or from

(20) $$\ell(R_1) = 0,$$

where R_1 is given by (18).

These two equations coincide for some $\beta_0 = \beta_0(\alpha,\ c)$.

(19) and (20) can be reduced to (10) and (11), respectively.

From (10) and (11) we get,

(21) $$r_1 = r_{\alpha,\beta,c} = ((2\alpha-\beta+c-1-D^{1/2})/(2\alpha-\beta+c+1-D^{1/2}))^{1/2},$$

where $D = 4(2-\alpha)h^2 - (2h+\beta+c)^2 + (\beta-2\alpha-c)^2$, and

(22) $r_2 = r_{\alpha,\beta,c}$

$$= (1+c)(1-\beta)/[((\beta(c+\alpha)+2(1-\alpha)-(c+1)\alpha) + ((\beta(c+\alpha)+2(1-\alpha)-(c+1)\alpha)^2 - (1+c)(1-\beta)(4\alpha^2-2\alpha\beta+2c\alpha-4\alpha+1-c+\beta-c\beta))^{1/2}],$$

respectively. To determine the $\beta_0 = \beta_0(\alpha)$ that makes the transition from (21) to (22) set $R_0 = R_1$.

It follows then

(23) $h^2 + ah = (2-\alpha)R_1^2$.

From (20), (16) and (23), we obtain

(24) $(3-2\alpha)a + 2\alpha - \beta + c = 2(2-\alpha)\rho$.

From (24), (10), we obtain

(25) $$a = \frac{(1-\alpha)\beta^2 + (3\alpha - 2c\alpha + 5c)\beta - (2\alpha^2 + 3c\alpha + c^2\alpha + 3\alpha - c^2 - 2)}{(1-\alpha)(2\alpha - \beta + c)}$$

Equation (12) is deduced from (10) and (25).

Note that $r_{\alpha,\beta,c} = r_1$ cannot be used when $\beta > \alpha$, since $D < 0$.

Therefore, $r_{\alpha,\beta,c}$ may be used for $\beta > \alpha$. In fact r_2 may be used for $\beta \geqq \alpha$ (see [3]).

Now we determine the form of the extremal functions $f_0(z)$ for Theorem. Taking into account (8) and the fact that the minimum in case (19) is realized at a point on the diameter $\eta = 0$, we conclude that $p(z)$ (see (6)) should in this case be taken in the form

$$p(z) = \frac{1}{2}\frac{1+ze^{-i\theta}}{1-ze^{-i\theta}} + \frac{1}{2}\frac{1+ze^{i\theta}}{1-ze^{i\theta}},$$

where $\cos\theta$ is found from the equation

$$h + (1-r_1^2)(1-2r_1\cos\theta + r_1^2)^{1} = R_0$$

in which the quantities r_1 and R_0 are determined by fomulars (21) and (17).

Then the extremal function $f_0(z)$ can be represented by the expression

$$f_0(z) = \frac{1}{c+1} z(c+1-2(c+\alpha)z\cos\theta + (c+2\alpha-1)z^2)(1-2z\cos\theta + z^2)^{\alpha-2}$$

In case (20), $p(z)$ is given by $p(z) = \frac{1+z}{1-z}$. Hence the extremal function

$$f_0(z) = \frac{1}{c+1} z(c+1-(c+2\alpha-1)z)(1-z)^{2\alpha-3}$$

The proof of Theorem is now completed.

This theorem was found by S. D. Bernardi [2] when $\alpha=\beta=0$ and c is positive integer.

References

1. H. S. Al-Amiri, On the radius of starlikeness of certain analytic functions, Proc. Amer. Math. Soc. 42 (1974), 466-474.
2. S. D. Bernardi, The radius of univalence of certain analytic functions, Proc. Amer. Math. Soc. 24 (1970), 312-318.
3. R. J. Libera and A. E. Livingston, On the univalence of some classes of regular functions, Proc. Amer. Math. Soc. 30 (1971), 327-336.
4. M. S. Robertson, Variational methods for functions with positive real part, Trans. Amer. Math. Soc. 102 (1962), 82-93.
5. V. A. Zmorovič, On bounds of convexity for starlike functions of order α in the circle $|z|<1$ and in the circular region $0<|z|<1$, Mat. Sb. 68 (1965), 518-526 ; English transl., Amer. Math. Soc. Transl. (2) 80 (1969), 203-213.